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A New General Method of Interpolation.

By Emory McClintock.

A general method of interpolation will be described below, which is apparently both easier to prove and easier to work than the general methods now in use. By general method, I mean a process applicable to all cases, including those in which the given values of the variable do not form an arithmetical progression. Before stating it, it will be well to recall briefly the two methods hitherto prevailing.

The first method, as we may call it for the sake of distinction, is known as that of Lagrange, though the credit of it has also been claimed for Euler. If we suppose that $\phi x_1, \phi x_2, \phi x_3, \ldots \phi x_k$ are given, where $x_1, x_2, \ldots x_k$ are any numbers, Lagrange's formula for determining ϕx, where x is any other number, is

$$\phi x = \mathrm{X}_1 \phi x_1 + \mathrm{X}_2 \phi x_2 + \mathrm{X}_3 \phi x_3 + \ldots + \mathrm{X}_k \phi x_k,$$

where

$$\mathrm{X}_1 = \frac{(x - x_2)(x - x_3) \ldots (x - x_k)}{(x_1 - x_2)(x_1 - x_3) \ldots (x_1 - x_k)},$$

$$\mathrm{X}_2 = \frac{(x - x_1)(x - x_3) \ldots (x - x_k)}{(x_2 - x_1)(x_2 - x_3) \ldots (x_2 - x_k)},$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\mathrm{X}_k = \frac{(x - x_1)(x - x_2) \ldots (x - x_{k-1})}{(x_k - x_1)(x_k - x_2) \ldots (x_k - x_{k-1})}.$$

The law by which the coefficients are formed is obvious, and the proof of the formula is not difficult.

The second method, that of Newton, is as follows, using the same notation as before. Let

$$\phi_1 x_2 = \frac{\varphi x_2 - \varphi x_1}{x_2 - x_1},$$

$$\phi_1 x_3 = \frac{\varphi x_3 - \varphi x_2}{x_3 - x_2},$$

and so on, the general form being

$$\phi_1 x_n = \frac{\varphi x_n - \varphi x_{n-1}}{x_n - x_{n-1}}.$$

The most convenient way of recording these divided differences is to set down, first, a column containing the numerical values of x_1, x_2, &c.; next, a column containing, opposite these numbers respectively, the values of ϕx_1, ϕx_2, &c.; and thirdly, a column containing $\phi_1 x_2$ opposite x_2, $\phi_1 x_3$ opposite x_3, and so on. Adjoining this column of differences, may be recorded a second series of differences, $\phi_2 x_3$, $\phi_2 x_4$, &c., calculated by the formula

$$\phi_2 x_n = \frac{\varphi_1 x_n - \varphi_1 x_{n-1}}{x_n - x_{n-2}};$$

then a third series,

$$\phi_3 x_n = \frac{\varphi_2 x_n - \varphi_2 x_{n-1}}{x_n - x_{n-3}},$$

and so on, the general rule being

$$\phi_{m+1} x_n = \frac{\varphi_m x_n - \varphi_m x_{n-1}}{x_n - x_{n-m-1}}.$$

Thus, if four terms have been given, we shall have this scheme,

x_n	ϕx_n	$\phi_1 x_n$	$\phi_2 x_n$	$\phi_3 x_n$
x_1	ϕx_1			
x_2	ϕx_2	$\phi_1 x_2$		
x_3	ϕx_3	$\phi_1 x_3$	$\phi_2 x_3$	
x_4	ϕx_4	$\phi_1 x_4$	$\phi_2 x_4$	$\phi_3 x_4$

To find ϕx, we have now to use this formula:

$$\phi x = \phi x_1 + (x - x_1)\, \phi_1 x_2 + (x - x_1)(x - x_2)\, \phi_2 x_3 + (x - x_1)(x - x_2)(x - x_3)\, \phi_3 x_4 + \dots .$$

The complete proof of this formula is laborious.*

Concerning the relative merits of these two methods, it may be said that if more than one term is to be interpolated the second is by far the best. If only one term is wanted, the matter is doubtful. Some writers give both methods; Boole, in doing so (*Finite Diffs.*, 2d ed.), shows a preference for the first method, Grunert for the second. Some writers, possibly to save space, mention only one of the two; Hymers gives the first, De Morgan the second. The first formula has often been commended for its elegance, exhibited both in the symmetry of its form and in the simplicity of its proof. The second, on the other hand, seems to be, on the whole, the easier to work in practice. After due examination, my conclusion is that, if all the multiplications and

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*See, for three different demonstrations, De Morgan, *Calculus*, pp. 550–552; Grunert, Supplement to Klügel's *Wörterbuch*, vol. II, pp. 43–46; Boucharlat, *Calcul*, 7th ed., pp. 417–420.

divisions are performed by logarithms, the first method can, adopting one or two obvious labor-saving devices, be worked with somewhat fewer references to the table of logarithms than the second, but that, on the other hand, it requires a more frequent use of some of the logarithms when found, and imposes a greater strain of attention; in short, that the second method is, for practical purposes, at least equal to the first. This, of course, is a matter which any one interested will examine and decide for himself.

The alternative method which I have to present resembles, and may be regarded, perhaps, as a substitute for, the second method. The formula is the same, the difference lying in the manner of obtaining the quantities $\phi_1 x_2$, $\phi_2 x_3$, &c. I form the several orders of divided differences according to this formula,

$$\phi_{m+1} x_n = \frac{\phi_m x_n - \phi_m x_{m+1}}{x_n - x_{m+1}}.$$

In applying this formula, ϕ_0 is to be considered equivalent to ϕ, the given terms being ϕx_1 or $\phi_0 x_1$, ϕx_2 or $\phi_0 x_2$, and so on. The computation of divided differences by this formula is simpler than by the second method, inasmuch as *the denominator, in all the orders, corresponds to the numerator.* The amount of labor required in both cases is the same, but the strain on the attention is lessened by employing a more symmetrical formula. The work done is more mechanical, and therefore easier. This statement may readily be tested.

The demonstration of the new method is so simple that the process might be said to prove itself. Since, from the equation last given,

$$\phi_m x_n = \phi_m x_{m+1} + (x_n - x_{m+1})\, \phi_{m+1} x_n,$$

we have successively, for $m = 0$, $m = 1$, &c.,

$$\phi_0 x_n = \phi_0 x_1 + (x_n - x_1)\, \phi_1 x_n,$$
$$\phi_1 x_n = \phi_1 x_2 + (x_n - x_2)\, \phi_2 x_n,$$
$$\phi_2 x_n = \phi_2 x_3 + (x_n - x_3)\, \phi_3 x_n,$$

and so on. Hence, by successive substitution,

$$\phi_0 x_n = \phi_0 x_1 + (x_n - x_1)\, \phi_1 x_2 + (x_n - x_1)(x_n - x_2)\, \phi_2 x_3 + (x_n - x_1)(x_n - x_2)$$
$$(x_n - x_3)\, \phi_3 x_4 + \ldots \text{ (k terms)}.$$

This general law is true, as proved, for all the given values of x_n, namely, x_1, x_2, $\ldots x_k$. Assuming the same law to be true of values not given, it becomes a convenient formula for interpolation. This assumption is the same in substance with those underlying the first and second methods, and the values determined in any given case by the three methods are identical. If we write x for x_n, and ϕ for ϕ_0, the formula becomes

$$\phi x = \phi x_1 + (x - x_1)\,\phi_1 x_2 + (x - x_1)(x - x_2)\,\phi_2 x_3 + \dots .$$

It may also be written in this form:

$$\phi x = \phi x_1 + (x - x_1)[\phi_1 x_2 + (x - x_2)\{\phi_2 x_3 + (x - x_3)[\phi_3 x_4 + \dots]\}].$$

If, for example, four terms are given, we may calculate successively

$$\psi_3 x_4 = (x - x_3)\,\phi_3 x_4,$$
$$\psi_2 x_3 = (x - x_2)\,(\phi_2 x_3 + \psi_3 x_4),$$
$$\psi_1 x_2 = (x - x_1)\,(\phi_1 x_2 + \psi_2 x_3),$$

and the general formula is reduced to this,

$$\phi x = \phi x_1 + \psi_1 x_2 .$$

Similarly, if five terms are given, we may begin by calculating

$$\psi_4 x_5 = (x - x_4)\,\phi_4 x_5 .$$

then

$$\psi_3 x_4 = (x - x_3)\,(\phi_3 x_4 + \psi_4 x_5),$$

and the rest as before. The same manner of lessening the labor may be followed for six or more given terms.

The results obtained by this method are neither more nor less accurate than those obtained by the preceding methods, being, as already stated, the same. In the following example, where seven values of a function of the sixth degree are given, the result is exactly correct.

x_n	$\phi_0 x_n$	$\phi_1 x_n$	$\phi_2 x_n$	$\phi_3 x_n$	$\phi_4 x_n$	$\phi_5 x_n$	$\phi_6 x_n$
$x_1 = 0$	5						
$x_2 = 1$	1	-4					
$x_3 = 4$	1	-1	1				
$x_4 = 5$	3	$-\frac{2}{5}$	$\frac{9}{10}$	$-\frac{1}{10}$			
$x_5 = 6$	5	0	$\frac{4}{5}$	$-\frac{1}{10}$	0		
$x_6 = 8$	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{8}$	$\frac{1}{120}$	$-\frac{1}{240}$	
$x_7 = 11$	-6	-1	$\frac{3}{10}$	$-\frac{1}{10}$	0	0	$\frac{1}{720}$

Each difference is derived from the one adjoining on the left, by subtracting from the latter the number at the top of the same column, and dividing the remainder by the corresponding difference taken from the first column. Let us suppose that $x = 2$; then, following the formulæ last indicated, we have, successively,

$$\begin{aligned}
\psi_6 x_7 &= (2-8)\,\frac{1}{720} = -\frac{1}{120}\,, \\
\psi_5 x_6 &= (2-6)\left(-\frac{1}{240} - \frac{1}{120}\right) = \frac{1}{20}\,, \\
\psi_4 x_5 &= (2-5)\left(0 + \frac{1}{20}\right) = -\frac{3}{20}\,, \\
\psi_3 x_4 &= (2-4)\left(-\frac{1}{10} - \frac{3}{20}\right) = \frac{1}{2}\,, \\
\psi_2 x_3 &= (2-1)\left(1 + \frac{1}{2}\right) = \frac{3}{2}\,, \\
\psi_1 x_2 &= (2-0)\left(-4 + \frac{3}{2}\right) = -5\,, \\
\phi x &= 5 - 5 = 0\,.
\end{aligned}$$

Note on the Demonstration of certain Formulæ for Interpolation.

When the given values of the variable are in arithmetical progression, interpolation is best performed by one of two well known formulæ, the choice between the two depending upon the circumstances of the case in hand. These formulæ are,

$$\phi\,(x+k) = \phi x + k\Delta\phi x + \frac{1}{2}\,k\,(k-1)\,\Delta^2\phi x + \ldots\,,$$

$$\phi\,(x+k) = \phi x + k\left[\Lambda\phi\left(x \pm \frac{1}{2}\right) + \frac{1}{2}\,(k \mp 1)\,\Lambda^2\phi x\right] + \frac{1}{2.3}\,k^{)3(}\left[\Lambda^3\phi\left(x \pm \frac{1}{2}\right) + \frac{1}{4}\,(k \mp 2)\,\Lambda^4\phi x\right] + \ldots\,.$$

The notation here used is that employed in my "Essay on the Calculus of Enlargement" (Amer. Journal, II, 101), namely,

$$k^{)3(} = (k+1)\,k\,(k-1)\,,$$
$$k^{)5(} = (k+2)(k+1)\,k\,(k-1)(k-2)\,,$$

and so on; also,

$$\Delta = \mathrm{E} - 1\,,$$
$$\Lambda = \mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}}\,,$$

where E is the symbol of Enlargement, E^h being an operation such that $\mathrm{E}^h\phi x = \phi\,(x+h)$, whence

$$\Delta\phi x = \phi\,(x+1) - \phi x\,,$$

$$\Lambda\phi x = \phi\left(x+\frac{1}{2}\right) - \phi\left(x-\frac{1}{2}\right),$$

$$\Lambda^2\phi x = \Lambda\phi\left(x+\frac{1}{2}\right) - \Lambda\phi\left(x-\frac{1}{2}\right),$$

and so on. The class of differences denoted by $\Lambda^n\phi x$ are called central differences. Of the two formulæ above stated, the first is a simple and obvious case of the Factorial Theorem given in the Essay referred to, p. 143. The second may be derived from the same theorem, as follows.

From the factorial theorem we have directly

$$\phi\mathrm{E} = \phi 1 + \phi\mathrm{E}0\,.\,\Lambda + \frac{1}{2}\,\phi\mathrm{E}0^{(2)}.\,\Lambda^2 + \frac{1}{2\,.\,3}\,\phi\mathrm{E}0^{(3)}.\,\Lambda^3 + \ldots\,, \qquad \text{(A)}$$

where $x^{(2)} = xx$, $x^{(3)} = x\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$, $x^{(4)} = x\,(x+1)\,x\,(x-1)$, $x^{(5)}$ $= x\left(x+\frac{3}{2}\right)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\left(x-\frac{3}{2}\right)$, and so on, the general form being

$$x^{(m)} = x\left(x+\frac{1}{2}\,m-1\right)\left(x+\frac{1}{2}\,m-2\right)\ldots\left(x-\frac{1}{2}\,m+1\right).$$

Also, in terms of mean central differences,

$$\phi\mathrm{E} = \phi 1\,.\,\mathrm{I} + \phi\mathrm{E}0\,.\,\mathrm{I}\Lambda + \frac{1}{2}\,\phi\mathrm{E}0^{)2(}.\,\mathrm{I}\Lambda^2 + \frac{1}{2\,.\,3}\,\phi\mathrm{E}0^{)3(}.\,\mathrm{I}\Lambda^3 + \ldots\,. \qquad \text{(B)}$$

Here,

$$\mathrm{I} = \frac{1}{2}\,(\mathrm{E}^{\frac{1}{2}} + \mathrm{E}^{-\frac{1}{2}}),$$

$$\mathrm{I}\Lambda^n\phi x = \frac{1}{2}\left(\Lambda^n\phi\left[x+\frac{1}{2}\right] + \Lambda^n\phi\left[x-\frac{1}{2}\right]\right);$$

also,

$$x^{)m(} = \left(x+\frac{1}{2}\,m-\frac{1}{2}\right)\left(x+\frac{1}{2}\,m-\frac{3}{2}\right)\ldots\left(x-\frac{1}{2}\,m+\frac{1}{2}\right).$$

Hence,

$$\Lambda = \Lambda 0\,.\,\mathrm{I}\Lambda + \frac{1}{2}\,\Lambda 0^{)2(}.\,\mathrm{I}\Lambda^2 + \frac{1}{2\,.\,3}\,\Lambda 0^{)3(}.\,\mathrm{I}\Lambda^3 + \ldots\,.$$

Since $\Lambda x^{)m(} = mx^{)m-1(}$, as was shown in the Essay referred to,

$$\Lambda = \mathrm{I}\Lambda + 0^{)1(}.\,\mathrm{I}\Lambda^2 + \frac{1}{2}\,0^{)2(}.\,\mathrm{I}\Lambda^3 + \frac{1}{2\,.\,3}\,0^{)3(}\mathrm{I}\Lambda^4 + \ldots\,.$$

But $0^{)1(} = 0$, $0^{)3(} = 0$, and so for all odd exponents; so that

$$\Lambda = \mathrm{I}\Lambda + \frac{1}{2}\, 0^{)2(} . \mathrm{I}\Lambda^3 + \frac{1}{2.3.4}\, 0^{)4(} . \mathrm{I}\Lambda^5 + \ldots,$$

and multiplying this by Λ^{n-1},

$$\Lambda^n = \mathrm{I}\Lambda^n + \frac{1}{2}\, 0^{)2(} . \mathrm{I}\Lambda^{n+2} + \frac{1}{2.3.4}\, 0^{)4(} . \mathrm{I}\Lambda^{n+4} + \ldots.$$

It follows that any even or odd central difference, or any expression composed of even or odd central differences, can be expressed, respectively, in even or odd mean central differences. Let

$$\phi_1\mathrm{E} = \phi 1 + \frac{1}{2}\, \phi\mathrm{E}0^{(2)} . \Lambda^2 + \frac{1}{2.3.4}\, \phi\mathrm{E}0^{(4)} . \Lambda^4 + \ldots,$$

$$\phi_2\mathrm{E} = \phi\mathrm{E}0 . \Lambda + \frac{1}{2.3}\, \phi\mathrm{E}0^{(3)} . \Lambda^3 + \ldots,$$

so that, observing equation (A), $\phi_1\mathrm{E} + \phi_2\mathrm{E} = \phi\mathrm{E}$. Then

$$\phi_1\mathrm{E} = a_0\mathrm{I} + a_2\mathrm{I}\Lambda^2 + a_4\mathrm{I}\Lambda^4 + \ldots,$$

$$\phi_2\mathrm{E} = a_1\mathrm{I}\Lambda + a_3\mathrm{I}\Lambda^3 + \ldots,$$

where a_0, a_2, &c., a_1, a_3, &c., are undetermined coefficients, and

$$\phi\mathrm{E} = a_0\mathrm{I} + a_1\mathrm{I}\Lambda + a_2\mathrm{I}\Lambda^2 + \ldots.$$

But, by (B),

$$\phi\mathrm{E} = \phi 1 . \mathrm{I} + \phi\mathrm{E}0 . \mathrm{I}\Lambda + \frac{1}{2}\, \phi\mathrm{E}0^{)2(} . \mathrm{I}\Lambda^2 + \ldots.$$

This determines the values of the coefficients, and it follows that the even or odd terms of (A) may be replaced, respectively, by the even or odd terms of (B). We derive in this way the following novel and important theorems in the Calculus of Enlargement, expressing $\phi\mathrm{E}$ in alternate central and mean central differences:

$$\phi\mathrm{E} = \phi 1 + \phi\mathrm{E}0 . \mathrm{I}\Lambda + \frac{1}{2}\, \phi\mathrm{E}0^{(2)} . \Lambda^2 + \frac{1}{2.3}\, \phi\mathrm{E}0^{)3(} . \mathrm{I}\Lambda^3 + \frac{1}{2.3.4}\, \phi\mathrm{E}0^{(4)} . \Lambda^4 + \ldots,$$

$$\phi\mathrm{E} = \phi 1 . \mathrm{I} + \phi\mathrm{E}0 . \Lambda + \frac{1}{2}\, \phi\mathrm{E}0^{)2(} . \mathrm{I}\Lambda^2 + \frac{1}{2.3}\, \phi\mathrm{E}0^{(3)} . \Lambda^3 + \frac{1}{2.3.4}\, \phi\mathrm{E}0^{)4(} . \mathrm{I}\Lambda^4 + \ldots.$$

These general theorems can, among other uses, be applied directly to the matter now in hand, interpolation. Let $\phi\mathrm{E} = \mathrm{E}^k$, and let ψx be the subject of operation; then

$$\psi(x + k) = \psi x + k\, \mathrm{I}\Lambda\psi x + \frac{1}{2}\, k^{(2)}\Lambda^2\psi x + \frac{1}{2.3}\, k^{)3(}\mathrm{I}\Lambda^3\psi x + \ldots,$$

$$\psi(x + k) = \mathrm{I}\psi x + k\Lambda\psi x + \frac{1}{2}\, k^{)2(}\mathrm{I}\Lambda^2\psi x + \frac{1}{2.3}\, k^{(3)}\Lambda^3\psi x + \ldots,$$

These formulæ are usually ascribed to Stirling, who published them in 1730. I have, however, seen the first and more important of the two quoted from Newton, 1711. They remained undemonstrated, it is said, until the discovery of the method of generating functions. The proof by that method, which may be found in Lacroix, is tedious and indirect, yet it is the only one I have met with, though other indirect demonstrations are certainly possible.*

Since $\mathrm{I} = \mathrm{E}^{\pm\frac{1}{2}} \mp \frac{1}{2}\Lambda$,

$$\mathrm{I}\Lambda^n\psi x = \Lambda^n\psi\left(x \pm \frac{1}{2}\right) \mp \frac{1}{2}\Lambda^{n+1}\psi x.$$

Making this substitution in the first formula just demonstrated, and observing that $k^{(n)} = k\,.\,k^{)n-1(}$, we derive the duplex formula which we set out to prove,

$$\psi(x+k) = \psi x + k\left[\Lambda\psi\left(x \pm \frac{1}{2}\right) + \frac{1}{2}(k \mp 1)\,\Lambda^2\psi x\right]$$
$$+ \frac{1}{2\,.\,3}k^{)3(}\left[\Lambda^3\psi\left(x \pm \frac{1}{2}\right) + \frac{1}{4}(k \mp 2)\,\Lambda^4\psi x\right] + \dots.$$

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* The difficulty does not lie in exhibiting the first few coefficients, but in proving their law. The first of these formulæ is said by Boole to have been discussed for Smith's Prize in 1860.